

THE MINKOWSKI $\varphi(x)$ FUNCTION AND SALEM'S PROBLEM

LA FONCTION $\varphi(x)$ DE MINKOWSKI ET PROBLÈME DE SALEM

GEDRIUS ALKAUSKAS

ABSTRACT. R. Salem (Trans. Amer. Math. Soc. 53 (3) (1943) 427-439) asked whether the Fourier-Stieltjes transform of the Minkowski question mark function $\varphi(x)$ vanishes at infinity. In this note we present several possible approaches towards the solution. For example, we show that this transform satisfies integral and discrete functional equations. Thus, we expect the affirmative answer to Salem's problem. In the end of this note we show that recent attempt to settle this question (S. Yakubovich, C. R. Acad. Sci. Paris, Ser. I **349** (11-12) (2011) 633-636) is fallacious.

RÉSUMÉ. R. Salem (Trans. Amer. Math. Soc. 53 (3) (1943) 427-439) demande si la transformée de Fourier-Stieltjes de la fonction point d'interrogation de Minkowski $\varphi(x)$ s'annule à l'infini. Dans cette note nous présentons plusieurs approches afin de résoudre cette question. Nous montrons par exemple que cette transformée satisfait des équations fonctionnelles discrètes et entières. Ainsi, nous conjecturons une réponse positive au problème de Salem. A la fin de cette note, nous montrons qu'une tentative récente pour répondre à cette question (S. Yakubovich, C. R. Acad. Sci. Paris, Ser. I **349** (11-12) (2011) 633-636) est en fait incorrecte.

Mathematical Analysis/Number Theory

1. SALEM'S PROBLEM

The Minkowski question mark function $\varphi(x) : [0, 1] \mapsto [0, 1]$ is defined by

$$\varphi([0, a_1, a_2, a_3, \dots]) = 2 \sum_{i=1}^{\infty} (-1)^{i+1} 2^{-\sum_{j=1}^i a_j}, \quad a_i \in \mathbb{N};$$

$x = [0, a_1, a_2, a_3, \dots]$ stands for the representation of x by a (regular) continued fraction. The function $\varphi(x)$ is continuous, strictly increasing, and singular. The extended Minkowski question mark function is defined by $F(x) = \varphi(\frac{x}{x+1})$, $x \in [0, \infty)$. Thus, for $x \in [0, 1]$, we have $\varphi(x) = 2F(x)$. The function $F(x)$ satisfies functional equations

$$2F(x) = \begin{cases} F(x-1) + 1 & \text{if } x \geq 1, \\ F(\frac{x}{1-x}) & \text{if } 0 \leq x < 1. \end{cases} \quad (1)$$

This implies $F(x) + F(1/x) = 1$. As was proved by Salem [2], the function $\varphi(x)$ satisfies Hölder condition of order $\alpha = (2 \log \frac{\sqrt{5}+1}{2})^{-1} \log 2 = 0.7202_+$. The Laplace-Stieltjes transform of $\varphi(x)$ is defined by [1]

$$\mathbf{m}(t) = \int_0^1 e^{xt} d\varphi(x), \quad t \in \mathbb{C}.$$

This is an entire function. The symmetry property $\varphi(x) + \varphi(1-x) = 1$ implies $\mathbf{m}(t) = e^t \mathbf{m}(-t)$. Let $d_n = \mathbf{m}(2\pi i n)$, $n \in \mathbb{N}$. Because of the symmetry property we have $d_n \in \mathbb{R}$, and thus

$$d_n = \int_0^1 \cos(2\pi n x) d\varphi(x), \quad n \in \mathbb{N}. \quad (2)$$

In 1943 Raphaël Salem [2] posed the following problem: prove or disprove that $d_n \rightarrow 0$, as $n \rightarrow \infty$. The question to determine whether Fourier transform of a given measure vanishes at infinity is a very delicate question whose answer depends on an intrinsic structure of this measure. There are various examples for both cases of behaviour [6]. As was noted in [2], the general theorem of Wiener [6] about Fourier-Stieltjes coefficients of continuous monotone functions with known modulus of continuity and the Cauchy-Schwartz inequality imply that $\sum_{n=1}^N |d_n| = O(N^{1-\alpha/2})$. Thus, $|d_n| \ll n^{-0.3601}$ on average. Via a partial summation and standard calculations we get that

$$\sum_{n=1}^{\infty} \frac{d_n}{n^{\sigma}} \text{ converges absolutely for } \sigma > 1 - \frac{\alpha}{2} = 0.6398_+, \text{ and } \varphi(x) - x = \sum_{n=1}^{\infty} \frac{d_n}{\pi n} \cdot \sin(2\pi n x), \quad x \in [0, 1]. \quad (3)$$

Note that $|\sum_{n=1}^N d_n| \leq 2 \int_0^1 |e^{2\pi i x} - 1|^{-1} d?(x)$ which is finite, since $?(x) = 1 - ?(1 - x) \asymp 2^{-1/x}$ as $x \rightarrow 0_+$. Thus, we inherit that the Dirichlet series $\sum_{n=1}^\infty d_n n^{-\sigma}$ converges (conditionally) for $\sigma > 0$.

The purpose of this note is to disseminate the knowledge of Salem's problem to a wider audience of mathematicians. We contribute to this topic with two new results. Vaguely speaking, they show that the coefficients d_n behave in the same manner as they behave "on average"; hence the answer to Salem's problem most likely is positive. Let, as usual, $J_\nu(\star)$ stand for the Bessel function with index ν .

Theorem 1. (Integral functional equation). *The function $\mathbf{m}(it)$ satisfies the following identity:*

$$\frac{i \mathbf{m}(is)}{2e^{2is} - e^{is}} = \int_0^\infty \mathbf{m}'(it) J_0(2\sqrt{st}) dt, \quad s > 0.$$

The integral is conditionally convergent.

Note that $\mathbf{m}(t)$ satisfies analogous integral equation on the negative real line [1]. Theorem 1, however, cannot be deduced from the latter by standard methods. If we formally pass to the limit $s \rightarrow \infty$ under the integral, the bound $|J_0(2\sqrt{st})| \ll (st)^{-1/4}$ would imply $\mathbf{m}(is) \rightarrow 0$. The same conclusion follows if we formally take the limit $s \rightarrow 0$. Unfortunately, this conditionally convergent integral cannot be dealt this way. In fact, let $\mathbf{n}(it) = e^{it}$. Then (7) shows that $i\mathbf{n}(is)e^{-2is} = \int_0^\infty \mathbf{n}'(it) J_0(2\sqrt{st}) dt$ for $s > 0$. Now the formal passage to the limit $s \rightarrow \infty$ gives the false result $e^{is} \rightarrow 0$. Therefore, if the solution of Salem's problem based on Theorem 1 is found, it should deal with the factor $(2e^{2is} - e^{is})^{-1}$, as opposed to e^{-2is} . The behaviour of $?(x)$ at $x = 0$ and $x = 1$ is of importance as well. Theorem 1 has a discrete analogue.

Theorem 2. (Discrete functional equation). *For any $m \in \mathbb{N}$ we have the following identity:*

$$d_m = \int_0^1 \cos\left(\frac{2\pi m}{x}\right) dx + 2 \sum_{n=1}^\infty d_n \cdot \int_0^1 \cos(2\pi n x) \cos\left(\frac{2\pi m}{x}\right) dx.$$

This sum is majorized by the series $Cm \sum_{n=1}^\infty |d_n| n^{-3/4}$ (see (3)) with an absolute constant C .

The theorem of Salem and Zygmund [6] shows that $d_n = o(1)$ implies $\mathbf{m}(it) = o(1)$; this is a general fact for the Fourier-Stieltjes transforms of non-decreasing functions. Another idea how to tackle Salem's problem is to approach it via the above system of infinite linear identities. This demands a detailed study of the integral $P(a, b) = \int_0^1 \cos(a/x + bx) dx$. Its exact asymptotics can be given in terms of elementary functions if $b > (1 + \epsilon)a$, or $b < (1 - \epsilon)a$ (b can be negative), or $b = a$, where $\epsilon > 0$ is fixed. The transition area $b \sim a$ exhibits a more complex behaviour. One can nevertheless give exact asymptotics in terms of Fresnel sine and cosine integrals, and this asymptotics is also valid in the transition area. These investigations are due to N. Temme [3]. Possibly, the full strength of these results can solve Salem's problem; our joint project with N. Temme is in progress.

2. THE PROOFS

Proof of Theorem 1. First, we will show that the integral does converge relatively. Indeed, let $X > 0$. Then

$$A(s, X) := \int_0^X \mathbf{m}'(it) J_0(2\sqrt{st}) dt = -i \int_0^X J_0(2\sqrt{st}) d\mathbf{m}(it) = -i J_0(2\sqrt{sX}) \mathbf{m}(iX) + i - i \int_0^X \mathbf{m}(it) J_1(2\sqrt{st}) \frac{s^{1/2}}{t^{1/2}} dt.$$

Let

$$\widehat{\mathbf{m}}(T) = \int_0^T \mathbf{m}(it) dt = \int_0^1 \frac{e^{ixT} - 1}{ix} d?(x). \text{ This implies } |\widehat{\mathbf{m}}(T)| \leq 2 \int_0^1 x^{-1} d?(x) = 5, \text{ for } T \geq 0. \quad (4)$$

We can continue:

$$\begin{aligned} A(s, X) &= -i J_0(2\sqrt{sX}) \mathbf{m}(iX) + i - i \int_0^X J_1(2\sqrt{st}) \cdot \frac{s^{1/2}}{t^{1/2}} d\widehat{\mathbf{m}}(t) \\ &= -i J_0(2\sqrt{sX}) \mathbf{m}(iX) + i - i J_1(2\sqrt{sX}) \frac{s^{1/2}}{X^{1/2}} \widehat{\mathbf{m}}(X) + \frac{i}{2} \int_0^X \widehat{\mathbf{m}}(t) \left(J_0(2\sqrt{st}) \frac{s}{t} - J_2(2\sqrt{st}) \frac{s}{t} - J_1(2\sqrt{st}) \frac{s^{1/2}}{t^{3/2}} \right) dt. \end{aligned} \quad (5)$$

The function under integral is bounded in the neighborhood of $t = 0$ since $\widehat{\mathbf{m}}(t)$ has a first order zero at $t = 0$, and for $\nu \in \mathbb{N}_0$, $J_\nu(u)$ has a zero of order ν at $u = 0$ (thus, no zero for $\nu = 0$). Further, we have the bound for the Bessel function $|J_\nu(u)| \ll u^{-1/2}$ as $u \rightarrow \infty$, ν is fixed. Thus, the function under integral is $\ll t^{-5/4}$ for $t > 1$, hence the integral (5) converges absolutely. Therefore there exists a finite limit $A(s) = \lim_{X \rightarrow \infty} A(s, X)$, and the integral in Theorem 1 converges conditionally. In fact, we used only the properties (4) and $|\mathbf{m}(it)| \leq 1$. Further, we take $X = \infty$ and substitute (4) into (5). We get

$$A(s) = i + \frac{i}{2} \int_0^1 \left[\int_0^\infty \frac{e^{ixt} - 1}{ix} d?(x) \right] \left(J_0(2\sqrt{st}) \frac{s}{t} - J_2(2\sqrt{st}) \frac{s}{t} - J_1(2\sqrt{st}) \frac{s^{1/2}}{t^{3/2}} \right) dt. \quad (6)$$

The double integral converges absolutely. Indeed, $(e^{ixt} - 1)(ix)^{-1} = \int_0^t e^{ixu} du \Rightarrow |(e^{ixt} - 1)(ix)^{-1}| \leq \min\{t, 2/x\}$. Now we easily obtain an absolute convergence: just use the bound t for $t \leq 1$ and the bound $2/x$ for $t \geq 1$. So Fubini's theorem allows us to interchange the order of integration in (6). After going backwards by integrating by parts, we get

$$A(s) = i + \frac{i}{2} \int_0^1 \left[\int_0^\infty (e^{ixt} - 1) \left(J_0(2\sqrt{st}) \frac{s}{t} - J_2(2\sqrt{st}) \frac{s}{t} - J_1(2\sqrt{st}) \frac{s^{1/2}}{t^{3/2}} \right) dt \right] \frac{d?(x)}{ix} = \int_0^1 \left[\int_0^\infty x e^{ixt} J_0(2\sqrt{st}) dt \right] d?(x).$$

For $x, s > 0$, we have the classical integral [4]

$$x \int_0^\infty e^{ixt} J_0(2\sqrt{st}) dt = i e^{-\frac{is}{x}}. \quad (7)$$

So, using $F(x) + F(1/x) = 1$, functional equations (1), and the symmetry property for $\mathbf{m}(t)$, we obtain

$$A(s) = 2i \int_0^1 e^{-\frac{is}{x}} dF(x) \stackrel{x \mapsto \frac{1}{x}}{=} 2i \int_1^\infty e^{-isx} dF(x) \stackrel{(1)}{=} 2i \int_0^1 \sum_{n=1}^\infty \frac{e^{-is(x+n)}}{2^n} dF(x) = \frac{i\mathbf{m}(-is)}{2e^{is} - 1} = \frac{i\mathbf{m}(is)}{2e^{2is} - e^{is}}. \quad \square$$

Proof of Theorem 2. We know that $?(x) - x$ can be expressed by the absolutely uniformly convergent series (3), which can be integrated term-by-term. Let, for $0 < \epsilon < 1$,

$$\widehat{A}(s, \epsilon) = i \int_\epsilon^1 e^{-\frac{is}{x}} d?(x) = 2i \int_1^{1/\epsilon} e^{-isx} dF(x).$$

We know that $\lim_{\epsilon \rightarrow 0+} \widehat{A}(s, \epsilon) = A(s)$. Thus,

$$\begin{aligned} \widehat{A}(s, \epsilon) &= i \int_\epsilon^1 e^{-\frac{is}{x}} d[(?(x) - x) + x] = i \int_\epsilon^1 e^{-\frac{is}{x}} dx - i e^{-\frac{is}{\epsilon}} (?(\epsilon) - \epsilon) + \int_\epsilon^1 (?(x) - x) e^{-\frac{is}{x}} \cdot \frac{s dx}{x^2} \\ &= i \int_\epsilon^1 e^{-\frac{is}{x}} dx - i e^{-\frac{is}{\epsilon}} (?(\epsilon) - \epsilon) + \sum_{n=1}^\infty \frac{d_n}{\pi n} \int_\epsilon^1 \sin(2\pi n x) e^{-\frac{is}{x}} \cdot \frac{s dx}{x^2} \\ &= i \int_\epsilon^1 e^{-\frac{is}{x}} dx - i e^{-\frac{is}{\epsilon}} (?(\epsilon) - \epsilon) + i \sum_{n=1}^\infty \frac{d_n}{\pi n} \sin(2\pi n \epsilon) e^{-\frac{is}{\epsilon}} + 2i \sum_{n=1}^\infty d_n \int_\epsilon^1 \cos(2\pi n x) e^{-\frac{is}{x}} dx. \end{aligned} \quad (8)$$

Take the imaginary part. Two series converge absolutely and uniformly with respect to ϵ . This follows from (3) and

Lemma. *Let $a > 0$. For a certain absolute constant $C > 1$ and any $\epsilon \in [0, 1]$ we have the following bound:*

$$\left| \int_\epsilon^1 \cos\left(bx + \frac{a}{x}\right) dx \right| < \begin{cases} C(a+1)b^{-3/4} & \text{if } b \geq 2\pi, \\ C(a+1)|b|^{-1} & \text{if } b \leq -2\pi. \end{cases}$$

If $2\pi \leq b \leq (C(a+1))^{4/3}$, the first bound is trivial since the integral is ≤ 1 . If $b > (C(a+1))^{4/3}$, we have the case $b > (1+\epsilon)a$ in the aforementioned cosine integral $P(a, b)$. In this case the asymptotics has only contributions from a stationary saddle point $x_0 = (a/b)^{1/2}$ (if the latter is in the neighborhood of ϵ) and the end point $x_1 = 1$. We deal with the case $b < 0$ similarly. Thus, the above bounds follow from results and techniques in [3]; the

exponent $3/4$ is the best possible and cannot be increased. This proves the Lemma. Note that $2 \cos(a/x) \cos(bx) = \cos(a/x + bx) + \cos(a/x - bx)$. Therefore for fixed s , the last series in (8) is majorized by $\hat{C}(s+1) \sum_{n=1}^{\infty} |d_n| n^{-3/4}$, and one can pass to the limit $\epsilon \rightarrow 0_+$ in (8) elementwise. This yields

$$\Re \frac{\mathfrak{m}(is)}{2e^{2is} - e^{is}} = \Im A(s) = \lim_{\epsilon \rightarrow 0_+} \Im \hat{A}(s, \epsilon) = \int_0^1 \cos\left(\frac{s}{x}\right) dx + 2 \sum_{n=1}^{\infty} d_n \cdot \int_0^1 \cos(2\pi n x) \cos\left(\frac{s}{x}\right) dx.$$

Now we finish with the substitution $s = 2\pi m$, $m \in \mathbb{N}$. \square

APPENDIX A.

The proof of Salem's problem presented in [5] is fallacious and cannot be fixed. Indeed, if we track down what properties of $?(x)$ are used in the proof, we find that the author only uses the fact that $?(x)$ is of bounded variation, that $?(0) = 0$, $?(1) = 1$, $\int_0^1 ?(x) x^{-5/4} dx < +\infty$, and $?(x) = 2F(x) = 2 + O(x^{-3})$ as $x \rightarrow +\infty$. Moreover, the last property is not needed as well, as we will now explain. The author writes $\int_0^1 = \int_0^\infty - \int_1^\infty$ (formula (18) in [5]), and derives asymptotic expansions of both integrals on the right side. But our main concern is still the asymptotics for the integral $\int_0^1 f(x) D(\tau, x) dx$, formula [5], (17). Now it is obvious that we can extend the function $f(x)$, initially defined for $[0, 1]$, to the interval $[1, \infty)$ in an almost arbitrary way. Eventually, if the asymptotic results we obtain are correct, two contributions from $[1, \infty)$ will annihilate one another. Nevertheless, the properties which were really used are obviously insufficient for the Fourier-Stieltjes transform of a singular measure to vanish at infinity; the Cantor "middle-third" distribution is a counterexample.

Now we will indicate where the mistake is. Assume that D. Naylor's result, which is the basis of authors results, is true ([5], formulas (8)-(9)). This would give the following consequence. Let $f(x)$ be continuous, $f(0) = 0$, $f(x) = O(x^b)$, as $x \rightarrow 0_+$ for a certain $b > 0$, and $f(x) = 0$ for $x \geq 1$. Then for a fixed f ,

$$\int_0^1 K_{i\tau}(x) f(x) \frac{dx}{x} = O(e^{-\pi\tau/2} \tau^{-N}) \text{ for every } N \geq 1, \text{ as } \tau \rightarrow \infty. \quad (9)$$

In fact, for $\tau > 0$ and any $x \in (0, 1]$ we have $K_{i\tau}(x) \sim -(2\pi)^{-1/2} e^{-\pi\tau/2} \tau^{-1/2} \sin\left[\tau \log\left(\frac{ex}{2\tau}\right)\right]$. One can extract further asymptotic terms which are of order τ^{-1} , τ^{-2} etc. smaller than the first one. Thus, direct calculation shows that the estimate (9) cannot hold for every such function $f(x)$ and $N = 2$ (just consider functions f such that $f(x) = 0$ for $x \in [0, 1/2]$). And indeed, Naylor's expansion works for $0 < \beta < \frac{\pi}{2}$. Meanwhile the case $\beta = \frac{\pi}{2}$, which is essential for the argument of [5] to work, is not allowed, and, moreover, this expansion in case $\beta = \frac{\pi}{2}$ is false, as we have just seen.

ACKNOWLEDGEMENTS

The author sincerely thanks Nico Temme for the help with the Lemma. This work was supported by the Lithuanian Science Council whose postdoctoral fellowship is being funded by European Union Structural Funds project "Postdoctoral Fellowship Implementation in Lithuania".

REFERENCES

- [1] G. Alkauskas, The moments of Minkowski question mark function: the dyadic period function, *Glasg. Math. J.* 52 (1) (2010), 41–64.
- [2] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.* 53 (3) (1943), 427–439.
- [3] N. Temme, Asymptotics of the integral $\int_0^1 \cos(a/x + bx) dx$ (preprint).
- [4] G. N. Watson, A treatise on the theory of Bessel function. Reprint of the second (1944) edition. Cambridge University Press, 1995.
- [5] S. Yakubovich, The Fourier-Stieltjes transform of Minkowski's $?(x)$ function and an affirmative answer to Salem's problem, *C. R. Acad. Sci. Paris, Ser. I* **349** (11-12) (2011) 633-636.
- [6] A. Zygmund, Trigonometric series. Vol. I, II. Reprint of the 1979 edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, LT-03225 Vilnius, Lithuania.
giedrius.alkauskas@gmail.com